DEPARTAMENTO DE INGENIERÍA MECÁNICA

# Diseño Computarizado

Método de elementos finitos: Elementos bidimensionales

Claudio García Herrera
Matías Inostroza Inostroza

Universidad de Santiago de Chile (USACH)
Facultad de Ingeniería - Departamento de Ingeniería Mecánica
Av. Bdo. O'Higgins 3363 - Santiago - CHILE
Correo: matias.inostroza.i@usach.cl y claudio.garcia@usach.cl
Laboratorio de biomecánica y biomateriales

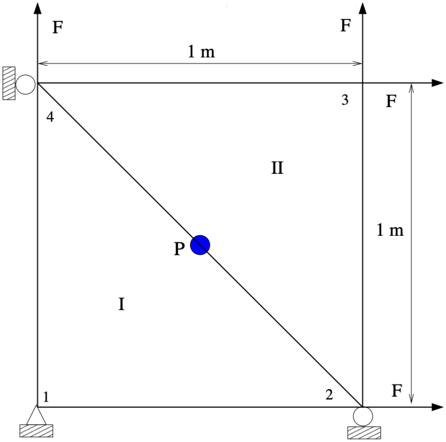
# Método de la rigidez



## Problema 1

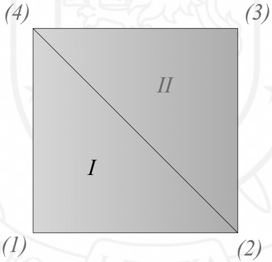
En la figura se tiene una placa cuadrada de acero isótropo ($E = 210$ GPa, $\nu = 0,3$) espesor $1$ mm, dicha placa se solicita con cargas de $F = 10$ kN cada una, produciendo un estado de esfuerzo biaxial plano en la placa. La placa se malla utilizando dos elementos triangulares, tal como se indica en la figura. Se pide:

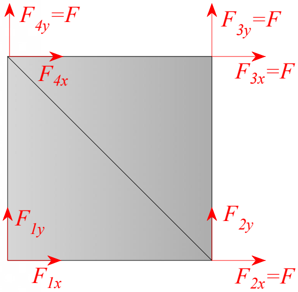
- Reacciones y desplazamientos en los nodos

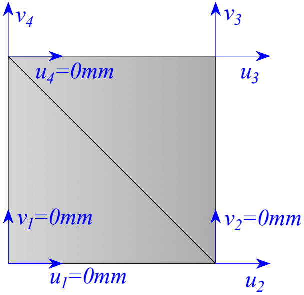


# Método de la rigidez

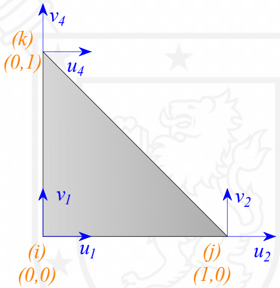
## Identificación de nodos y elementos



# Matriz de rigidez elemento I: Tensión plana




$$\boldsymbol{B} = \frac{1}{1}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}\frac{1}{m}$$

$$\boldsymbol{D} = \frac{210\cdot 10^9}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} Pa$$

$$\boldsymbol{K} = \int_V \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}dV \rightarrow \boldsymbol{K} = tA\boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}$$

$b_i = y_j - y_k = -1 c_i = x_k - x_j = -1$

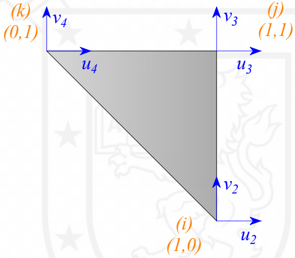
$b_j = y_k - y_i = 1\ c_j = x_i - x_k = 0$

$b_k = y_i - y_j = 0\ c_k = x_j - x_i = 1$

- $A = \frac{1}{2}1\cdot 1 = 0.5m^2$
- $t = 0.001m$
- $\Delta = 2A = 1m^2$

$$\boldsymbol{K}_I = \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} 15.58 & 7.5 & -11.54 & -4.04 & -4.04 & -3.46 \\ 7.5 & 15.58 & -3.46 & -4.04 & -4.04 & -11.54 \\ -11.54 & -3.46 & 11.54 & 0.0 & 0.0 & 3.46 \\ -4.04 & -4.04 & 0.0 & 4.04 & 4.04 & 0.0 \\ -4.04 & -4.04 & 0.0 & 4.04 & 4.04 & 0.0 \\ -3.46 & -11.54 & 3.46 & 0.0 & 0.0 & 11.54 \end{bmatrix} \end{array} \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_4 \\ v_4 \end{matrix} \; 10^7\frac{N}{m}$$

# Matriz de rigidez elemento II: Tensión plana




$$\boldsymbol{B} = \frac{1}{1}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \frac{1}{m}$$

$$\boldsymbol{D} = \frac{210 \cdot 10^9}{1 - 0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} Pa$$

$$\boldsymbol{K} = \int_V \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} dV \rightarrow \boldsymbol{K} = tA\boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}$$

$b_i = y_j - y_k = 0$ $c_i = x_k - x_j = -1$

$b_j = y_k - y_i = 1$ $c_j = x_i - x_k = 1$

$b_k = y_i - y_j = -1$ $c_k = x_j - x_i = 0$

- $A = \frac{1}{2} 1 \cdot 1 = 0.5m^2$
- $t = 0.001m$
- $\Delta = 2A = 1m^2$

$$\boldsymbol{K}_{II} = \begin{array}{c} \begin{array}{cccccc} u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 4.04 & 0.0 & -4.04 & -4.04 & 0.0 & 4.04 \\ 0.0 & 11.54 & -3.46 & -11.54 & 3.46 & 0.0 \\ -4.04 & -3.46 & 15.58 & 7.5 & -11.54 & -4.04 \\ -4.04 & -11.54 & 7.5 & 15.58 & -3.46 & -4.04 \\ 0.0 & 3.46 & -11.54 & -3.46 & 11.54 & 0.0 \\ 4.04 & 0.0 & -4.04 & -4.04 & 0.0 & 4.04 \end{bmatrix} \end{array} \begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array} 10^7 \frac{N}{m}$$

# Ensamblaje de la matriz global: $K_l$

$$
\begin{array}{c}
\begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_4 & v_4 \end{array} \\
K_l = \begin{bmatrix}
15.58 & 7.5 & -11.54 & -4.04 & -4.04 & -3.46 \\
7.5 & 15.58 & -3.46 & -4.04 & -4.04 & -11.54 \\
-11.54 & -3.46 & 11.54 & 0.0 & 0.0 & 3.46 \\
-4.04 & -4.04 & 0.0 & 4.04 & 4.04 & 0.0 \\
-4.04 & -4.04 & 0.0 & 4.04 & 4.04 & 0.0 \\
-3.46 & -11.54 & 3.46 & 0.0 & 0.0 & 11.54
\end{bmatrix}
\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_4 \\ v_4 \end{array} 10^7
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\
K_g = \begin{bmatrix}
15.58 & 7.5 & -11.54 & -4.04 & 0.0 & 0.0 & -4.04 & -3.46 \\
7.5 & 15.58 & -3.46 & -4.04 & 0.0 & 0.0 & -4.04 & -11.54 \\
-11.54 & -3.46 & 11.54 & 0.0 & 0.0 & 0.0 & 0.0 & 3.46 \\
-4.04 & -4.04 & 0.0 & 4.04 & 0.0 & 0.0 & 4.04 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-4.04 & -4.04 & 0.0 & 4.04 & 0.0 & 0.0 & 4.04 & 0.0 \\
-3.46 & -11.54 & 3.46 & 0.0 & 0.0 & 0.0 & 0.0 & 11.54
\end{bmatrix}
\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array} 10^7 \frac{N}{m}
\end{array}
$$

# Ensamblaje de la matriz global: $K_I$

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$$K_{II} = \begin{array}{c} \begin{matrix} u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} 4.04 & 0.0 & -4.04 & -4.04 & 0.0 & 4.04 \\ 0.0 & 11.54 & -3.46 & -11.54 & 3.46 & 0.0 \\ -4.04 & -3.46 & 15.58 & 7.5 & -11.54 & -4.04 \\ -4.04 & -11.54 & 7.5 & 15.58 & -3.46 & -4.04 \\ 0.0 & 3.46 & -11.54 & -3.46 & 11.54 & 0.0 \\ 4.04 & 0.0 & -4.04 & -4.04 & 0.0 & 4.04 \end{bmatrix} \end{array} \begin{matrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{matrix} \; 10^7$$

$$K_g = \begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{matrix} \\ \begin{bmatrix} 15.58 & 7.5 & -11.54 & -4.04 & 0.0 & 0.0 & -4.04 & -3.46 \\ 7.5 & 15.58 & -3.46 & -4.04 & 0.0 & 0.0 & -4.04 & -11.54 \\ -11.54 & -3.46 & 11.54 + 4.04 & 0.0 + 0.0 & -4.04 & -4.04 & 0.0 + 0.0 & 3.46 + 4.04 \\ -4.04 & -4.04 & 0.0 + 0.0 & 4.04 + 11.54 & -3.46 & -11.54 & 4.04 + 3.46 & 0.0 + 0.0 \\ 0.0 & 0.0 & -4.04 & -3.46 & 15.58 & 7.5 & -11.54 & -4.04 \\ 0.0 & 0.0 & -4.04 & -11.54 & 7.5 & 15.58 & -3.46 & -4.04 \\ -4.04 & -4.04 & 0.0 + 0.0 & 4.04 + 3.46 & -11.54 & -3.46 & 4.04 + 11.54 & 0.0 + 0.0 \\ -3.46 & -11.54 & 3.46 + 4.04 & 0.0 + 0.0 & -4.04 & -4.04 & 0.0 + 0.0 & 11.54 + 4.04 \end{bmatrix} \end{array} \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{matrix} \; 10^7 \frac{N}{m}$$

## Matriz global ensamblada

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$$K_g = \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array}$$

$$K_g = \begin{bmatrix} 15.58 & 7.5 & -11.54 & -4.04 & 0.0 & 0.0 & -4.04 & -3.46 \\ 7.5 & 15.58 & -3.46 & -4.04 & 0.0 & 0.0 & -4.04 & -11.54 \\ -11.54 & -3.46 & 15.58 & 0.0 & -4.04 & -4.04 & 0.0 & 7.5 \\ -4.04 & -4.04 & 0.0 & 15.58 & -3.46 & -11.54 & 7.5 & 0.0 \\ 0.0 & 0.0 & -4.04 & -3.46 & 15.58 & 7.5 & -11.54 & -4.04 \\ 0.0 & 0.0 & -4.04 & -11.54 & 7.5 & 15.58 & -3.46 & -4.04 \\ -4.04 & -4.04 & 0.0 & 7.5 & -11.54 & -3.46 & 15.58 & 0.0 \\ -3.46 & -11.54 & 7.5 & 0.0 & -4.04 & -4.04 & 0.0 & 15.58 \end{bmatrix} 10^7 \frac{N}{m}$$

# Método de la rigidez



$$
K_g = \begin{array}{c} \begin{matrix} \quad u_1 & \quad v_1 & \quad u_2 & \quad v_2 & \quad u_3 & \quad v_3 & \quad u_4 & \quad v_4 \end{matrix} \\ \begin{bmatrix} 15.58 & 7.5 & -11.54 & -4.04 & 0.0 & 0.0 & -4.04 & -3.46 \\ 7.5 & 15.58 & -3.46 & -4.04 & 0.0 & 0.0 & -4.04 & -11.54 \\ -11.54 & -3.46 & 15.58 & 0.0 & -4.04 & -4.04 & 0.0 & 7.5 \\ -4.04 & -4.04 & 0.0 & 15.58 & -3.46 & -11.54 & 7.5 & 0.0 \\ 0.0 & 0.0 & -4.04 & -3.46 & 15.58 & 7.5 & -11.54 & -4.04 \\ 0.0 & 0.0 & -4.04 & -11.54 & 7.5 & 15.58 & -3.46 & -4.04 \\ -4.04 & -4.04 & 0.0 & 7.5 & -11.54 & -3.46 & 15.58 & 0.0 \\ -3.46 & -11.54 & 7.5 & 0.0 & -4.04 & -4.04 & 0.0 & 15.58 \end{bmatrix} \end{array} 10^7 \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{bmatrix}
$$

# GDL ordenados

$$\begin{bmatrix} \begin{bmatrix} 15.58 & -4.04 & -4.04 & 7.5 \\ -4.04 & 15.58 & 7.5 & -4.04 \\ -4.04 & 7.5 & 15.58 & -4.04 \\ 7.5 & -4.04 & -4.04 & 15.58 \end{bmatrix} & \begin{bmatrix} -11.54 & -3.46 & 0.0 & 0.0 \\ 0.0 & 0.0 & -3.46 & -11.54 \\ 0.0 & 0.0 & -11.54 & -3.46 \\ -3.46 & -11.54 & 0.0 & 0.0 \end{bmatrix} \\ \begin{bmatrix} -11.54 & 0.0 & 0.0 & -3.46 \\ -3.46 & 0.0 & 0.0 & -11.54 \\ 0.0 & -3.46 & -11.54 & 0.0 \\ 0.0 & -11.54 & -3.46 & 0.0 \end{bmatrix} & \begin{bmatrix} 15.58 & 7.5 & -4.04 & -4.04 \\ 7.5 & 15.58 & -4.04 & -4.04 \\ -4.04 & -4.04 & 15.58 & 7.5 \\ -4.04 & -4.04 & 7.5 & 15.58 \end{bmatrix} \end{bmatrix} 10^7 \begin{bmatrix} u_2 \\ u_3 \\ v_3 \\ v_4 \\ u_1 \\ v_1 \\ v_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} F_{2x} \\ F_{3x} \\ F_{3y} \\ F_{4y} \\ F_{1x} \\ F_{1y} \\ F_{2y} \\ F_{4x} \end{bmatrix}$$

Ordenamiento de tipo:

$$\begin{bmatrix} [K_{aa}] & [K_{ab}] \\ [K_{ba}] & [K_{bb}] \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_b \end{bmatrix} = \begin{bmatrix} F_a \\ F_b \end{bmatrix}$$

# GDL ordenados

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$$\begin{bmatrix} \begin{bmatrix} 15.58 & -4.04 & -4.04 & 7.5 \\ -4.04 & 15.58 & 7.5 & -4.04 \\ -4.04 & 7.5 & 15.58 & -4.04 \\ 7.5 & -4.04 & -4.04 & 15.58 \end{bmatrix} & \begin{bmatrix} -11.54 & -3.46 & 0.0 & 0.0 \\ 0.0 & 0.0 & -3.46 & -11.54 \\ 0.0 & 0.0 & -11.54 & -3.46 \\ -3.46 & -11.54 & 0.0 & 0.0 \end{bmatrix} \\ \begin{bmatrix} -11.54 & 0.0 & 0.0 & -3.46 \\ -3.46 & 0.0 & 0.0 & -11.54 \\ 0.0 & -3.46 & -11.54 & 0.0 \\ 0.0 & -11.54 & -3.46 & 0.0 \end{bmatrix} & \begin{bmatrix} 15.58 & 7.5 & -4.04 & -4.04 \\ 7.5 & 15.58 & -4.04 & -4.04 \\ -4.04 & -4.04 & 15.58 & 7.5 \\ -4.04 & -4.04 & 7.5 & 15.58 \end{bmatrix} \end{bmatrix} 10^7 \begin{bmatrix} u_2 \\ u_3 \\ v_3 \\ v_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10000 \\ 10000 \\ 10000 \\ 10000 \\ F_{1x} \\ F_{1y} \\ F_{2y} \\ F_{4x} \end{bmatrix}$$

# Resultados

En ausencia de tensiones residuales, efectos térmicos, y ya que todos los desplazamiento conocidos son cero:

$$\begin{bmatrix} 15.58 & -4.04 & -4.04 & 7.5 \\ -4.04 & 15.58 & 7.5 & -4.04 \\ -4.04 & 7.5 & 15.58 & -4.04 \\ 7.5 & -4.04 & -4.04 & 15.58 \end{bmatrix} 10^7 \begin{bmatrix} u_2 \\ u_3 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 10000 \\ 10000 \\ 10000 \\ 10000 \end{bmatrix}$$

$$u_2 = u_3 = v_3 = v_4 = 6.67 \cdot 10^{-5} m = 0.067 mm$$

Y para las fuerzas:

$$\begin{bmatrix} -11.54 & 0.0 & 0.0 & -3.46 \\ -3.46 & 0.0 & 0.0 & -11.54 \\ 0.0 & -3.46 & -11.54 & 0.0 \\ 0.0 & -11.54 & -3.46 & 0.0 \end{bmatrix} 10^7 \begin{bmatrix} u_2 \\ u_3 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2y} \\ F_{4x} \end{bmatrix} \Rightarrow \begin{bmatrix} F_{1x} = -10000 \\ F_{1y} = -10000 \\ F_{2y} = -10000 \\ F_{4x} = -10000 \end{bmatrix} N$$

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Laboratorio de biomecánica y biomateriales