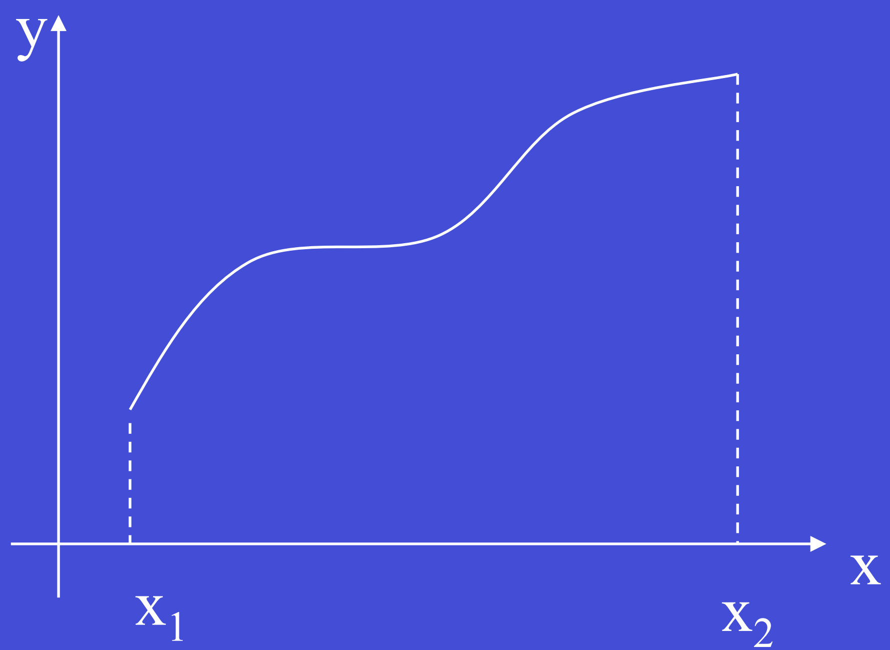
Métodos de Integración

$$I = \int_{x_1}^{x_2} f(x)\,dx$$



Newton-Cotes
Cuadratura Gaussiana

Aplicaciones

- Areas
- Volúmenes
- Longitudes
- Momentos de inercia
- Trabajo y Energía
- Calor
- Flujos
- Corriente
- Entalpía
- etc

Método de Newton-Cotes

$$I = \int_{x_0}^{x_n} f(x)\,dx$$

$$I \approx \int_{x_0}^{x_n} p_n(x)\,dx$$

$$f(x) \approx p_n(x)$$

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

Pasos:

1.- Se divide el dominio [a,b] en n intervalos equidistantes

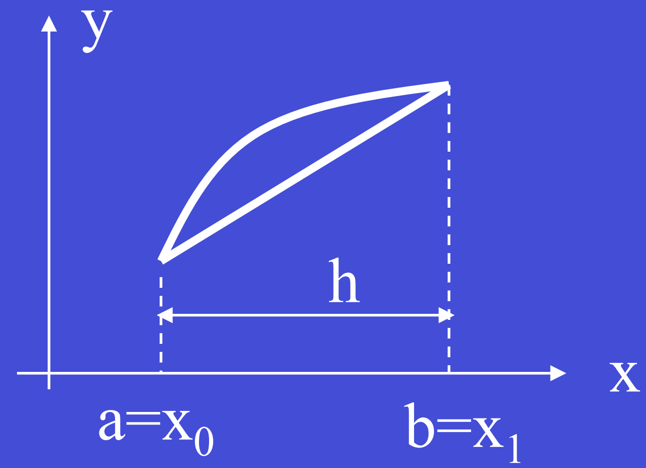
2.- Se aproxima f(x) por un polinomio de grado n, $p_n(x)$

3.- Se integra I

Casos particulares:

Método trapezoidal

$$f(x) \approx p_1(x)$$
$$p_1(x) = a_0 + a_1 x$$



Se determinan los coeficientes a_0 y a_1 a partir de los puntos dados

$P_1(x_0)$ $\quad y_0 = a_0 + a_1 x_0$

$P_1(x_1)$ $\quad y_1 = a_0 + a_1 x_1$

$$\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} y_0 \\ y_1 \end{Bmatrix}$$

$$a_0 = \frac{\begin{vmatrix} y_0 & x_0 \\ y_1 & x_1 \end{vmatrix}}{\begin{vmatrix} 1 & x_0 \\ 1 & x_1 \end{vmatrix}} = \frac{y_0 x_1 - y_1 x_0}{x_1 - x_0}$$

$$a_1 = \frac{\begin{vmatrix} 1 & y_0 \\ 1 & y_1 \end{vmatrix}}{\begin{vmatrix} 1 & x_0 \\ 1 & x_1 \end{vmatrix}} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$p_1(x) = a_0 + a_1 x$$

$$p_1(x) = \frac{y_0 x_1 - y_1 x_0}{x_1 - x_0} + \frac{y_1 - y_0}{x_1 - x_0} x$$

$$I \approx \int_{x_0}^{x_1} p_1(x)\,dx = \int_{x_0}^{x_1} \left(\frac{y_0 x_1 - y_1 x_0}{x_1 - x_0} + \frac{y_1 - y_0}{x_1 - x_0} x \right) dx$$

$$I \approx \int_{x_0}^{x_1} \left(\frac{y_0 x_1 - y_1 x_0}{x_1 - x_0} + \frac{y_1 - y_0}{x_1 - x_0} x \right) dx$$

$$I \approx \frac{y_0 x_1 - y_1 x_0}{x_1 - x_0}(x_1 - x_0) + \frac{1}{2}\frac{y_1 - y_0}{x_1 - x_0}(x_1^2 - x_0^2)$$

$$I \approx y_0 x_1 - y_1 x_0 + \frac{1}{2}(y_1 - y_0)(x_1 + x_0)$$

$$I \approx y_0 x_1 - y_1 x_0 + \frac{1}{2}(y_1 x_1 + y_1 x_0 - y_0 x_1 - y_0 x_0)$$

$$I \approx \frac{1}{2}(y_0 x_1 - y_1 x_0 + y_1 x_1 - y_0 x_0)$$

$$I \approx \frac{1}{2}(x_1 - x_0)(y_1 + y_0)$$

MÉTODO TRAPEZOIDAL

$$I \approx \frac{1}{2}(x_1 - x_0)(y_1 + y_0)$$

$$h = (x_1 - x_0) \qquad y_1 = f(x_1) \qquad y_0 = f(x_0)$$

$$I \approx \frac{h}{2}(f(x_0) + f(x_1))$$

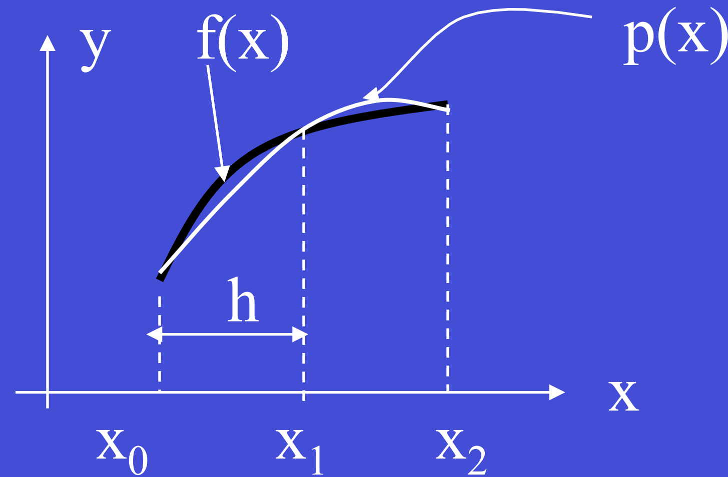
$$I = \frac{h}{2}(f(x_0) + f(x_1)) - \frac{h^3}{12} f''(\varepsilon)$$

Casos particulares:

Método de Simpson

$$f(x) \approx p_2(x)$$

$$p_2(x) = a_0 + a_1 x + a_2 x^2$$



Se determinan los coeficientes a_0 , a_1 y a_2 a partir de los puntos dados

$P_2(x_0)$	$y_0 = a_0 + a_1 x_0 + a_2 x_0^2$
$P_2(x_1)$	$y_1 = a_0 + a_1 x_1 + a_2 x_1^2$
$P_2(x_2)$	$y_2 = a_0 + a_1 x_2 + a_2 x_2^2$

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

METODO Simpson 1/3

$$I \approx \int_{x_0}^{x_2} p_2(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

METODO General N puntos

N=3

$$I \approx \int_{x_0}^{x_3} p_3(x)dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

N=4

$$I \approx \int_{x_0}^{x_4} p_4(x)dx = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$$

Métodos compuestos de Integración
Se divide el intervalo [a,b] en subintervalos equidistantes

Trapezoidal compuesto

$$I = \int_a^b f(x)dx$$

$$I \approx \int_{a=x_0}^{x_1} q_1(x)dx + \int_{x_1}^{x_2} q_2(x)dx + \int_{x_2}^{x_3} q_3(x)dx \cdots\cdots + \int_{x_{n-1}}^{x_n=b} q_n(x)dx$$

$$\frac{x_1 - x_0}{2}[f(x_0) + f(x_1)] \qquad \frac{x_2 - x_1}{2}[f(x_1) + f(x_2)] \qquad \frac{x_n - x_{n-1}}{2}[f(x_{n-1}) + f(x_n)]$$

$$I \approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots\cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$$I \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

Simpson compuesto

$$I \approx \frac{h}{3}\left[f(x_0) + 4\sum_{i=1}^{n-1} f(x_i) + 2\sum_{j=3}^{n-2} f(x_j) + f(x_n) \right]$$

$$i = 2,4,6,8 \ldots\ldots \qquad j = 3,5,7,9 \ldots\ldots$$

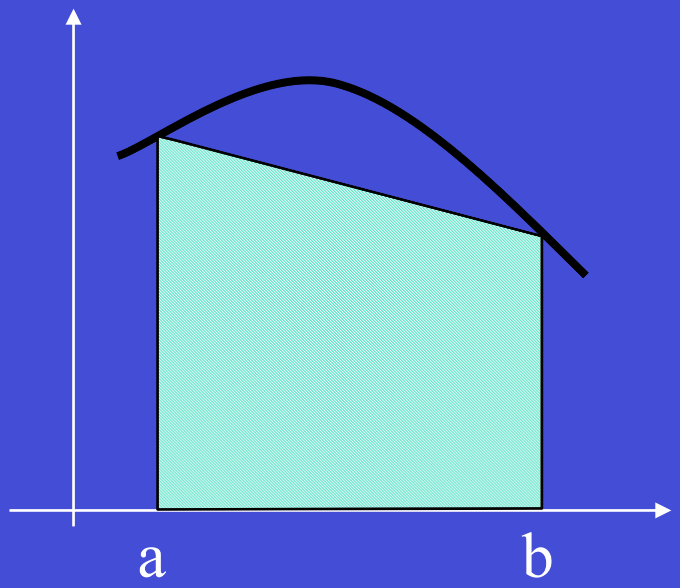
Ejemplo:

Encuentre la integral aproximada de la función:

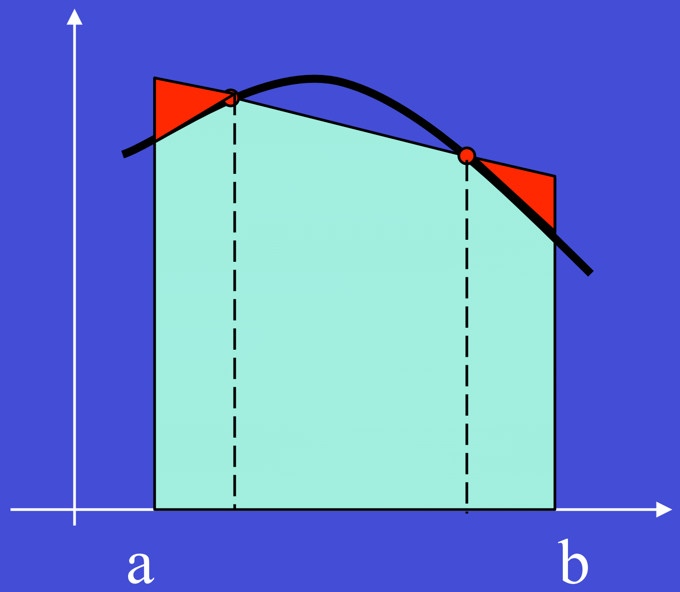
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$I \approx \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Integración Cuadratura Gaussiana:



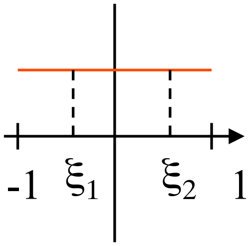
Trapezoidal



Cuadratura Gaussiana

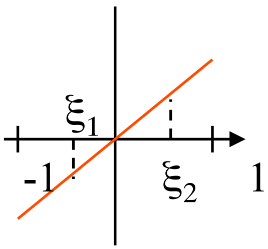
Determinación de coeficientes y puntos de integración, tomamos 2 puntos

$$I \approx c_1 f(x_1) + c_2 f(x_2)$$



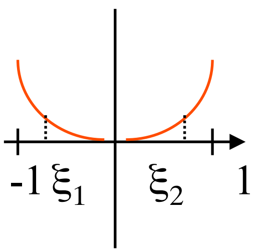
$$f(\xi) = f_0 \qquad I = \int_{-1}^{1} f_0 d\xi = 2 f_0$$

$$I \approx H_1 f(\xi_1) + H_2 f(\xi_2) \qquad I \approx H_1 f_0 + H_2 f_0 \qquad 2 = H_1 + H_2$$



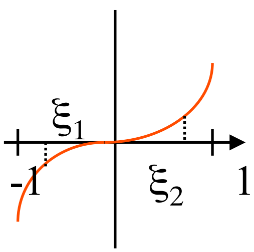
$$f(\xi) = \xi \qquad I = \int_{-1}^{1} \xi d\xi = \frac{1}{2}\xi^2 \Big|_{-1}^{1} = 0$$

$$I \approx H_1 f(\xi_1) + H_2 f(\xi_2) \qquad I \approx H_1 \xi_1 + H_2 \xi_2 \qquad 0 = H_1 \xi_1 + H_2 \xi_2$$



$$f(\xi) = \xi^2 \qquad I = \int_{-1}^{1} \xi^2 d\xi = \frac{1}{3}\xi^3 \Big|_{-1}^{1} = \frac{2}{3}$$

$$I \approx H_1 \xi_1^{\ 2} + H_2 \xi_2^{\ 2} \qquad I \approx H_1 \xi_1^{\ 2} + H_2 \xi_2^{\ 2} \qquad \frac{2}{3} = H_1 \xi_1^{\ 2} + H_2 \xi_2^{\ 2}$$



$$f(\xi) = \xi^3 \qquad I = \int_{-1}^{1} \xi^3 d\xi = \frac{1}{4}\xi^4 \Big|_{-1}^{1} = 0$$

$$I \approx H_1 \xi_1^{\ 3} + H_2 \xi_2^{\ 3} \qquad I \approx H_1 \xi_1^{\ 3} + H_2 \xi_2^{\ 3} \qquad 0 = H_1 \xi_1^{\ 3} + H_2 \xi_2^{\ 3}$$

$$2 \approx H_1 + H_2$$

$$0 \approx H_1\xi_1 + H_2\xi_1 \Rightarrow 0 \approx -H_1 + H_2 \text{ (obs: } \xi_1 = -1)$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\frac{2}{3} \approx H_1\xi_1^{\,2} + H_2\xi_2^{\,2} \Rightarrow \xi_1^{\,2} + \xi_2^{\,2} = \frac{2}{3}$$

$$0 \approx H_1\xi_1^{\,3} + H_2\xi_2^{\,3} \Rightarrow \xi_1^{\,3} + \xi_2^{\,3} = 0$$

$$\xi_1^{\,3} = \xi_2^{\,3} \Rightarrow \xi_1 = \pm\xi_2$$

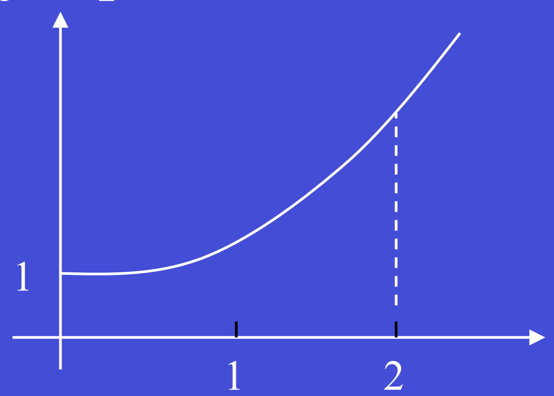
$$\xi_1^{\,2} + \xi_1^{\,2} = \frac{2}{3} \qquad \Rightarrow \xi_1^{\,2} = \frac{1}{3} \qquad \Rightarrow \xi_1 = -\sqrt{\frac{1}{3}} \qquad \Rightarrow \xi_2 = \sqrt{\frac{1}{3}}$$

$$I \approx c_1 f(x_1) + c_2 f(x_2)$$

$$I \approx \sum_{i=1}^{n} c_i f(x_i)$$

Ptos Gauss	Coeficientes c_i	Puntos de Precisión x_i
2	1.0	+/- 0.5773502
3	0.888 0.555 0.555	+/- 0.7745966 0.0000000
4	0.3478548 0.6521451	+/- 0.8611363 +/- 0.3399810
5	0.2369268 0.4786286 0.5688888	+/- 0.9061798 +/- 0.5384693 0.000000000

Ejemplo:



$$I = \int_0^2 e^y \, dy$$

I(exacta)=6.3890

Para dos puntos se tiene

$C_1 = C_2 = 1$ $\qquad x_1 = -0.5773502$ $\qquad x_2 = 0.5773502$

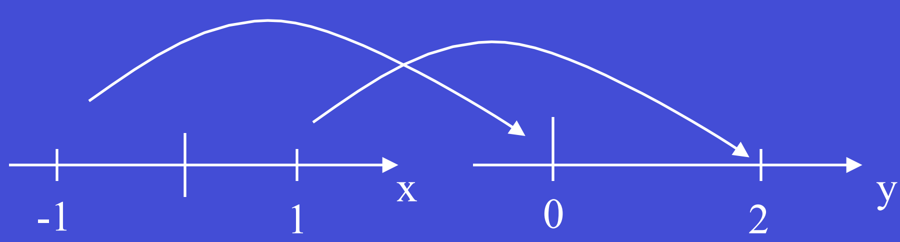
Cambio de variable lineal:

y=ax+b

Condiciones de borde

0 = a (-1) +b

2 = a (1) +b $\qquad$ a=1 b=1



y = x + 1 dy=dx

$$I = \int_{-1}^{1} e^{x+1} dx$$

$$I \approx e^{x_1+1} + e^{x_2+1}$$

$$I \approx e^{0.4226498} + e^{1.5773502}$$

$$I = 1.525999797 + 4.842108179$$

$$I = 6.36810797$$

$$\varepsilon = 0.327$$

Integración en varias dimensiones

$$I = \iint f(x,y)\,dxdy$$

$$I = \int_{-1}^{1}\left[\int_{-1}^{1} f(x,Y)\,dx\right]dy$$

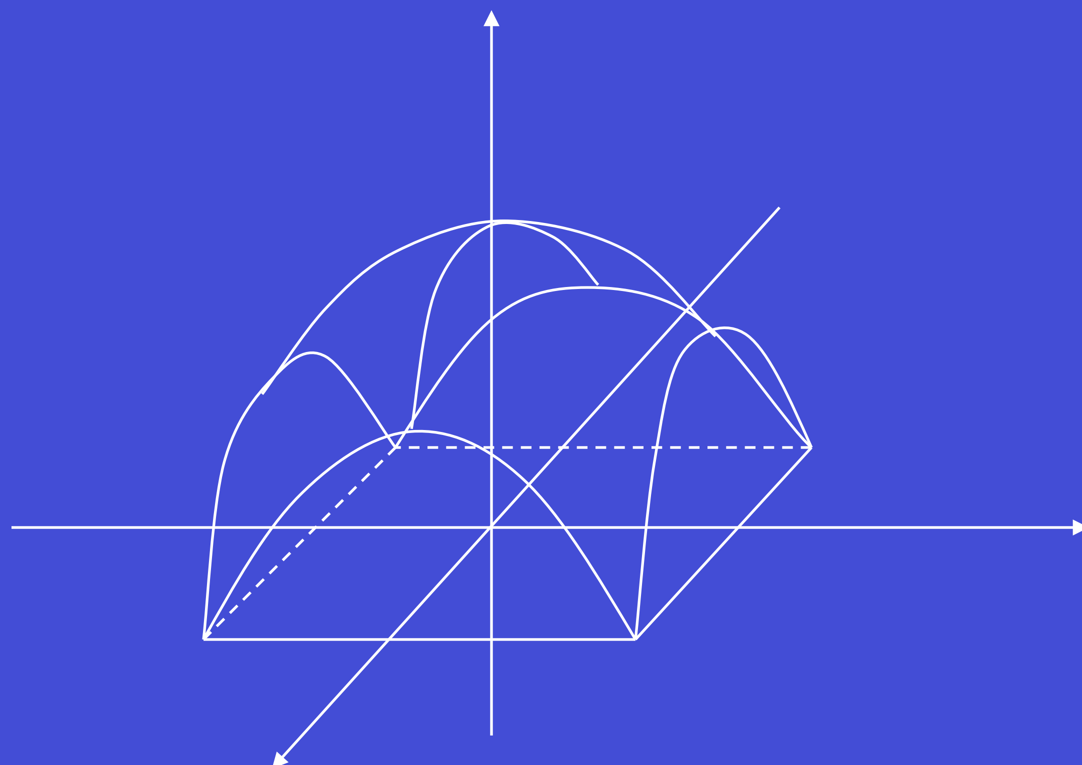
$$I \approx \int_{-1}^{1}[c_1 f(x_1,Y) + c_2 f(x_2,Y)]\,dy$$

$$I \approx \int_{-1}^{1} c_1 f(x_1,y)\,dy + \int_{-1}^{1} c_2 f(x_2,y)\,dy$$

$$I \approx c_1[c_1 f(x_1,y_1) + c_2 f(x_1,y_2)] + c_2[c_1 f(x_2,y_1) + c_2 f(x_2,y_2)]$$

$$I \approx c_1 c_1 f(x_1,y_1) + c_1 c_2 f(x_1,y_2) + c_2 c_1 f(x_2,y_1) + c_2 c_2 f(x_2,y_2)$$

Ejemplo:



Solución:

Al tener una base de 2 m de lado se toma como dominio los valores:

$$-1 \le x \le 1 \qquad -1 \le y \le 1$$

Como estamos en el dominio adecuado, se toman los puntos de precisión directamente, es decir:

$$x = \pm \frac{1}{\sqrt{3}} \qquad y = \pm \frac{1}{\sqrt{3}}$$

La función de la semi esfera esta dada por:

$$f(x, y) = \sqrt{2 - x^2 - y^2}$$

$$I \approx c_1 c_1 f(x_1, y_1) + c_1 c_2 f(x_1, y_2) + c_2 c_1 f(x_2, y_1) + c_2 c_2 f(x_2, y_2)$$

$$I \approx 4\sqrt{2 - x^2 - y^2} = 4.619 \qquad e = 1.6\%$$